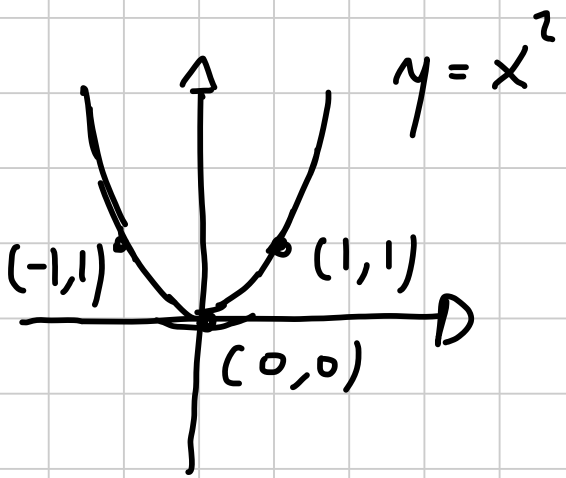
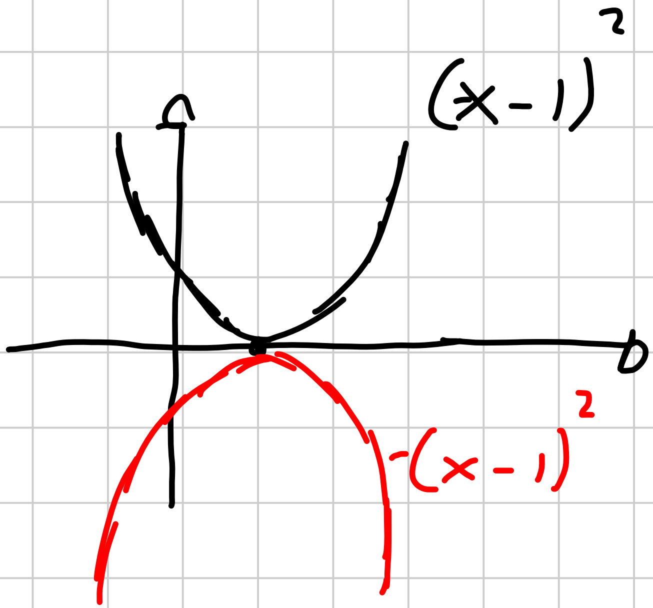
1) Si scriva la funzione integrale di:

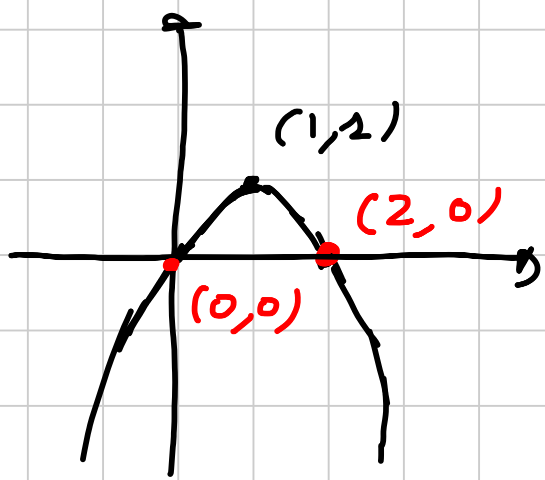
retta

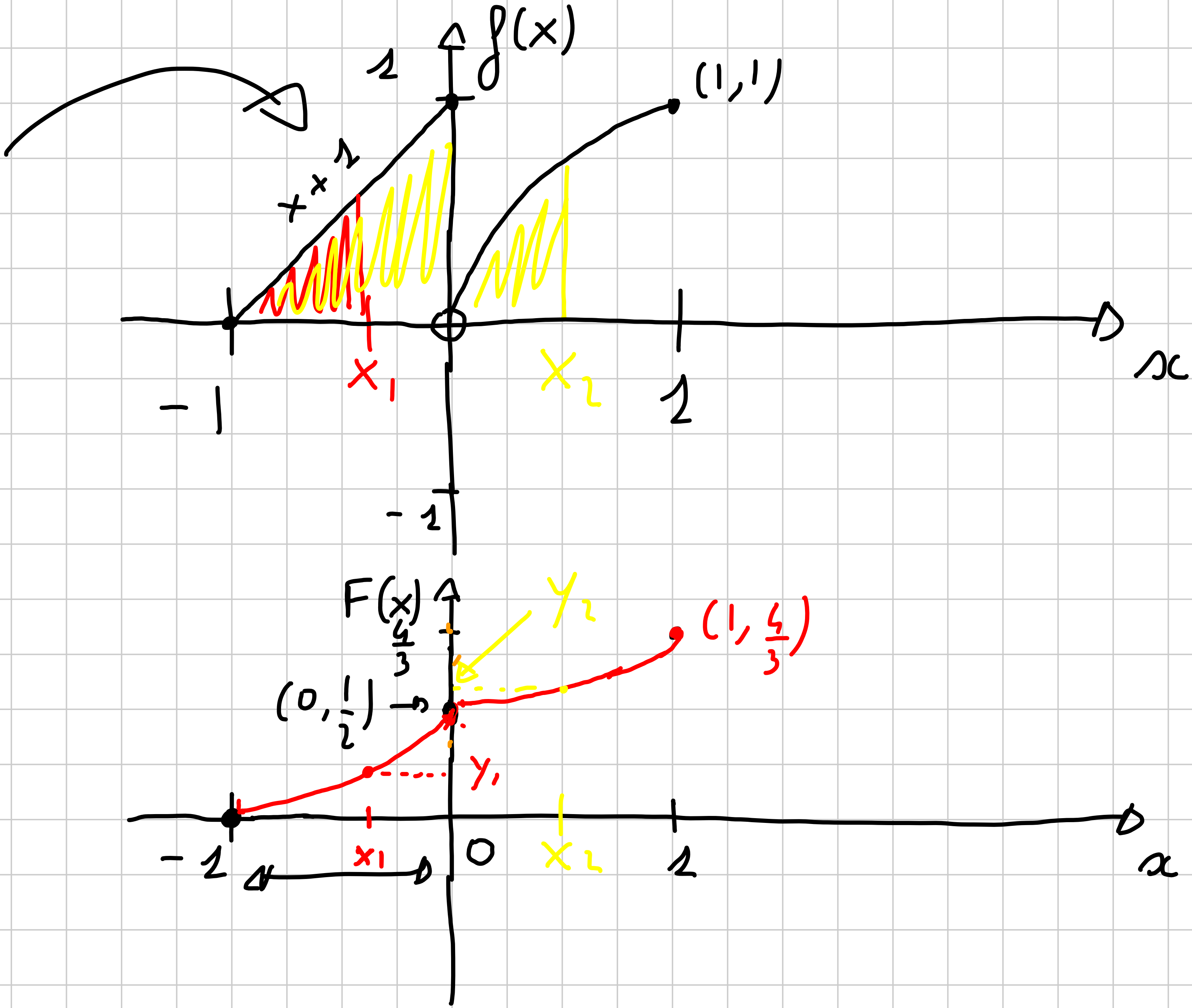
$$f(x) = \begin{cases} x+1 & -1 \le x \le 0 \\ 1-(x-1)^2 & 0 < x \le 1 \end{cases}$$

nell' intervallo $[-1, 1]$









f(x)
1
(1,1)
x+1
X1
X2
-1
1
x
-1
F(x)
4/3
Y2
(1, 4/3)
(0, 1/2)
Y1
-1
x1
0
X2
1
x

$f(x) \in [a, b] \qquad F(x) = \int_a^x f(t)\,dt$

per $x \in [-1, 0]$

$$F(x) = \int_{-1}^{x} f(t)\,dt = \int_{-1}^{x} (t+1)\,dt =$$

$$\left[\frac{t^2}{2} + t\right]_{-1}^{x} = \frac{x^2}{2} + x - \left(\frac{1}{2} - 1\right) = \frac{x^2}{2} + x + \frac{1}{2}$$

per $x \in [-1, 0] \qquad F(x) = \frac{x^2}{2} + x + \frac{1}{2}$

$x = -1 \quad F(x) = ?$ $\quad (-1, 0)$

$$F(-1) = \frac{(-1)^2}{2} - 1 + \frac{1}{2} = 0$$

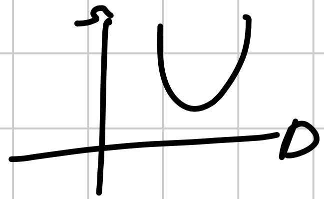
$x = 0 \quad F(x) = ?$ $\quad \left(0, \frac{1}{2}\right)$

$$F(0) = \frac{0^2}{2} + 0 + \frac{1}{2} = \frac{1}{2}$$

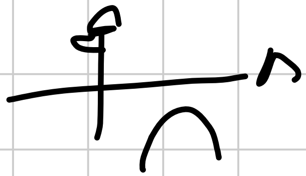
$$F(x) = \frac{x^2}{2} + x + \frac{1}{2}$$

$f(x) = ax^2 + bx + c$

$a > 0$



$a < 0$



per $x \in \,]0,1]$

$$F(x) = \int_{-1}^{x} f(t)\,dt = \int_{-1}^{0} f(t)\,dt + \int_{0}^{x} f(t)\,dt$$

punto

$$= \int_{-1}^{0} (t+1)\,dt + \int_{0}^{x} \left[1-(t-1)^2\right] dt =$$

$$= \left[\frac{t^2}{2} + t\right]_{-1}^{0} + \int_{0}^{x} (\cancel{1} - t^2 + 2t - \cancel{1})\,dt$$

$$= \left[\frac{0^2}{2} + 0 - \left(\frac{(-1)^2}{2} - 1\right)\right] + \left[-\frac{t^3}{3} + \frac{\cancel{2}t^2}{\cancel{2}}\right]_0^x$$

$$= \frac{1}{2} + \left[-\frac{x^3}{3} + x^2 - 0 \right] = -\frac{x^3}{3} + x^2 + \frac{1}{2}$$

per $x \in]0,1]$ $\quad F(x) = -\frac{x^3}{3} + x^2 + \frac{1}{2}$

$x = 0 \quad F(x) = ?$ $\qquad \left(0, \frac{1}{2}\right)$

$$F(0) = \frac{1}{2}$$

$x = 1 \quad F(x) = ?$ $\qquad \left(1, \frac{4}{3}\right)$

$$F(1) = -\frac{1^3}{3} + 1^2 + \frac{1}{2} = \frac{4}{3}$$

$$F(x) = -\frac{x^3}{3} + x^2 + \frac{1}{2}$$

$$F'(x) = -\frac{3x^2}{3} + 2x = -x^2 + 2x$$

$$F''(x) = -2x + 2$$

$$F''(x) \geq 0 \qquad -2x + 2 \geq 0$$

$$-2x \geq -2$$

$$x \leq 1$$

		1	
$F''(x)$	$+$		$-$
$F(x)$	$\cup$		$\cap$

$$\int_0^x \left[1 - (t-1)^2\right] dt = \int_0^x 1\, dt - \int_0^x (t-1)^2 dt =$$

$$= \left[t\right]_0^x - \int_0^x f'(t) f(t)^2 dt = (x - 0) - \left[\frac{(t-1)^3}{3}\right]_0^x =$$

$$= x - \left(\frac{(x-1)^3}{3} - \underbrace{\frac{(0-1)^3}{3}}_{-\frac{1}{3}} \right) = x - \left(\frac{(x-1)^3}{3} + \frac{1}{3} \right)$$

$$= \frac{3x - (x-1)^3 + 1}{3} = \frac{3x - (x^3 - 3x^2 + 3x - 1) + 1}{3} =$$

$$= \frac{\cancel{3x} - x^3 + 3x^2 - \cancel{3x} \cancel{+1} \cancel{+1}}{3} = -\frac{x^3}{3} + x^2 + \cancel{\frac{2}{3}}$$

1) $\displaystyle\int_0^{+\infty} \frac{1}{(x+1)^2}\,dx = \lim_{a \to +\infty} \int_0^{a} \frac{1}{(x+1)^2}\,dx = (*)$

$$\int \frac{1}{(x+1)^2}\,dx = \int 1(x+1)^{-2}\,dx = \int f'(x)\cdot f(x)^{-2}\,dx =$$

$f(x) = x+1$

$f'(x) = 1$

$$= \frac{(x+1)^{-2+1}}{-2+1} = -(x+1)^{-1} = -\frac{1}{x+1}$$

$$(*) = \lim_{a \to +\infty}\left[-\frac{1}{x+1}\right]_0^a = \lim_{a \to +\infty}\left[-\frac{1}{a+1} - \left(-\frac{1}{0+1}\right)\right] =$$

$$= \lim_{a \to +\infty} \left[-\frac{1}{a+1} + 1 \right] = 1$$

$$\frac{1}{a+1} \to 0$$

2) $$\int_e^{+\infty} \frac{1}{x \ln^3 x}\, dx = \lim_{a \to +\infty} \int_e^a \frac{1}{x \ln^3 x}\, dx =$$

$$\int \frac{1}{x \ln^3 x}\, dx = \int \frac{1}{x} \cdot \ln^{-3} x \, dx = \int f'(x) \cdot f(x)^{-3}\, dx =$$

$$\ln^2 x = (\ln x)^2$$

$$f(x) = \ln x$$
$$f'(x) = \frac{1}{x}$$

$$= \frac{(\ln x)^{-3+1}}{-3+1} = \frac{\ln^{-2} x}{-2}$$

$$= -\frac{1}{2 \ln^2 x}$$

$$= \lim_{a \to +\infty} \left[-\frac{1}{2\ln^2 x} \right]_e^a = \lim_{a \to +\infty} \left[-\frac{1}{2\ln^2 a} - \left(-\frac{1}{2\ln^2 e} \right) \right] =$$

$$= \lim_{a \to +\infty} \left[-\frac{1}{2\ln^2 a} + \frac{1}{2\ln^2 e} \right] = \lim_{a \to +\infty} \left[-\frac{1}{2\ln^2 a} + \frac{1}{2} \right]$$

$\ln e = 1$ $\qquad \to 0$

$$= \frac{1}{2}$$

3) $$\int_{-\infty}^{0} x e^x \, dx = \lim_{b \to -\infty} \int_b^0 x e^x \, dx = (*)$$

$$\underbrace{\int x e^x \, dx} = x e^x - \int e^x \, dx = x e^x - e^x$$

$$\int \underbrace{f'(x)\,g(x)} = f(x)\,g(x) - \int f(x)\,g'(x)$$

$f'(x) = e^x \qquad f(x) = e^x$

$g(x) = x \qquad g'(x) = 1$

$$(*) = \lim_{b \to -\infty} \Big[xe^x - e^x \Big]_b^0 =$$

$$\lim_{b \to -\infty} \Big[0e^0 - e^0 - \left(be^b - e^b \right) \Big] =$$

$$\lim_{b \to -\infty} \Big[0 - 1 - be^b + e^b \Big] =$$

$$\lim_{b \to -\infty} \left[\underbrace{e^{b}}_{0} - \overset{\infty \cdot 0}{b e^{b}} - 1 \right] = \lim_{b \to -\infty} \left(\underbrace{\underbrace{-b e^{b}}_{+\infty \cdot 0}}_{0} - 1 \right) = -1$$

$$\lim_{b \to -\infty} -b e^{b} = \lim_{b \to -\infty} \frac{\overset{+\infty}{-b}}{\underbrace{e^{-b}}_{+\infty}} = 0$$

$$= \lim_{t \to +\infty} \frac{t}{e^{t}} = 0$$

c. v $\quad -b = t$

$b \to -\infty \to t \to +\infty$

4) $$\int_{-\infty}^{+\infty} \frac{x}{1+x^{2}} \, dx = \lim_{\substack{a \to +\infty \\ b \to -\infty}} \int_{b}^{a} \frac{x}{1+x^{2}} \, dx = (*)$$

$$\frac{1}{2}\int \frac{2x}{1+x^2}\,dx = \frac{1}{2}\int \frac{f'(x)}{f(x)}\,dx = \frac{1}{2}\ln|f(x)| =$$

$f(x) = 1+x^2$

$f'(x) = 2x$

$$= \frac{1}{2}\ln|1+x^2| = \frac{1}{2}\ln(1+x^2)$$

$$(*) = \lim_{\substack{a\to+\infty \\ b\to-\infty}} \left[\frac{1}{2}\ln(1+x^2)\right]_b^a =$$

$$= \lim_{\substack{a\to+\infty \\ b\to-\infty}} \left[\underbrace{\frac{1}{2}\ln(\underbrace{1+a^2}_{+\infty})}_{+\infty} - \underbrace{\frac{1}{2}\ln(\underbrace{1+b^2}_{+\infty})}_{\substack{+\infty \\ -\infty}}\right] = ?$$

1) $\displaystyle\int_0^{+\infty} \frac{x}{1+x^2}\,dx$ diverge

$\dfrac{x}{1+x^2}$ $\quad D: \forall x \in \mathbb{R}$

$$\lim_{x \to \infty} \frac{x}{1+x^2} = \lim_{x \to +\infty} \frac{x}{x^2} = \lim_{x \to +\infty} \frac{1}{x}$$

$$x \to +\infty \quad \frac{x}{1+x^2} \sim \frac{1}{x}$$ diverge

$$\left[\int_1^{+\infty} \frac{1}{x^\alpha}\,dx \ \begin{cases} \text{converge} & \alpha \geq 2 \\ \text{diverge} & 0 < \alpha \leq 1 \end{cases} \right]$$

2) $\displaystyle\int_1^{+\infty} \frac{1}{x^2+x^3}\,dx$ converge

$x \to +\infty \quad \dfrac{1}{x^2+x^3} \sim \dfrac{1}{x^3} \to$ converge

$$\lim_{x\to+\infty} \frac{1}{x^3+x^2} = \lim_{x\to+\infty} \frac{1}{x^3}$$

3) $\displaystyle\int_0^{+\infty} \frac{1}{\sqrt{x^3+x+1}}\,dx$ converge

$$\lim_{x\to+\infty} \frac{1}{\sqrt{x^3+x+1}} = \lim_{x\to+\infty} \frac{1}{\sqrt{x^3\left(1+\cancel{\frac{1}{x^2}}+\cancel{\frac{1}{x^3}}\right)}} =$$

$$= \lim_{x \to +\infty} \frac{1}{\sqrt{x^3}} = \lim_{x \to +\infty} \frac{1}{x^{\frac{3}{2}}}$$

$$x \to +\infty \quad \frac{1}{\sqrt{x^3+x+1}} \sim \frac{1}{x^{\frac{3}{2}}}$$

converge

$\alpha > 1$

4) $$\int_1^{+\infty} \frac{1}{x+\ln x}\, dx$$

diverge

$$\lim_{x \to +\infty} \frac{1}{\underset{+\infty + \infty}{x+\ln x}} = \lim_{x \to +\infty} \frac{1}{x\left(1+\cancel{\frac{\ln x}{x}}\right)} = \lim_{x \to +\infty} \frac{1}{x}$$

$$\frac{1}{x+\ln x} \sim \frac{1}{x}$$ — diverge

5) $$\int_{-\infty}^{+\infty} \frac{1}{x^2+e^x}\,dx$$

converge

- $f \leq g$ e g converge $\Rightarrow$ f converge
- $f \geq g$ e g diverge $\Rightarrow$ f diverge

$$f = \frac{1}{x^2+e^x} < \frac{1}{x^2}$$ $\Rightarrow$ converge

$$x^2+e^x > x^2$$

converge

$\cancel{2}$ $5 > 2$

$\frac{1}{5} < \frac{1}{2}$

6) $\displaystyle\int_{-\infty}^{+\infty} \frac{1}{x^2+|x|+1}\,dx$ converge

$\sim \dfrac{1}{x^2}$ converge

$x \to +\infty \quad \dfrac{1}{x^2+|x|+1} = \dfrac{1}{x^2+x+1}$

$$\lim_{x \to +\infty} \underbrace{\frac{1}{x^2+x+1}} = \lim \frac{1}{x^2\left(1+\underset{0}{\cancel{\frac{1}{x}}}+\underset{0}{\cancel{\frac{1}{x^2}}}\right)} = \underbrace{\frac{1}{x^2}}$$

$$\frac{1}{x^2+e^x} \sim \frac{1}{e^x}$$