Calcola l'integrale usando la definizione:

1) $$\int_0^{+\infty} \frac{1}{e^x}\,dx = \lim_{a \to +\infty} \int_0^{a} \frac{1}{e^x}\,dx =$$

$$\int \frac{1}{e^x}\,dx = \int e^{-x}\,dx = -e^{-x}$$

$$= \lim_{a \to +\infty} \left[-e^{-x}\right]_0^a = \lim_{a \to +\infty} \left[-e^{-a} - \left(-e^{-0}\right)\right] =$$

$$= \lim_{a \to +\infty} \left[\underbrace{-e^{-a}}_{0} + 1\right] = 1$$

$$2)\quad \int_{-\infty}^{+\infty} \frac{x}{1+x^2}\,dx = \underbrace{A)\int_{-\infty}^{0} \frac{x}{1+x^2}\,dx} + \underbrace{B)\int_{0}^{+\infty} \frac{x}{1+x^2}\,dx}$$

$$A)\quad \lim_{b\to-\infty} \int_{b}^{0} \frac{x}{1+x^2}\,dx = \lim_{b\to-\infty}\left[\frac{1}{2}\ln(1+x^2)\right]_b^0 =$$

$$\frac{1}{2}\int \frac{2x}{1+x^2}\,dx = \frac{1}{2}\int \frac{f'(x)}{f(x)}\,dx = \frac{1}{2}\ln|1+x^2|$$

$$f(x) = 1+x^2 \qquad\qquad = \frac{1}{2}\ln(1+x^2)$$

$$f'(x) = 2x$$

$$= \lim_{b\to-\infty}\left[\cancel{\frac{1}{2}\ln(1+0)} - \frac{1}{2}\ln(1+b^2)\right] = \lim_{b\to-\infty}\left[-\frac{1}{2}\ln(1+b^2)\right]$$

$$-\infty$$

quindi $\lim \nexists \Rightarrow$

B) $\displaystyle \lim_{b \to +\infty} \int_0^a \frac{x}{1+x^2}\,dx$

$$\int_{-\infty}^{+\infty} \frac{x}{1+x^2}\,dx = \underbrace{\lim_{b\to-\infty} \int_b^0 \frac{x}{1+x^2}\,dx}_{\text{diverge}} + \lim_{a\to+\infty} \int_0^a \frac{x}{1+x^2}\,dx$$

SERIE

Serie geometrica $\sum_{n=0}^{\infty} r^n$

$-1 < r < 1$

$|r| < 1$ converge $\rightarrow \dfrac{1}{1-r}$

$r \geq 1$ diverge

$r \leq -1$ indeterminata

Calcolare la somma delle seguenti serie

1) $\sum_{n=0}^{\infty} 2^{(1-2n)} = \sum_{n=0}^{\infty} 2^{1} \cdot 2^{-2n} = \sum_{n=0}^{\infty} 2^{1} \cdot \left(\frac{1}{2^2}\right)^{n} =$

$= 2 \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^{n} = 2 \cdot \boxed{} = 2 \cdot \frac{4}{3} = \frac{8}{3}$

$$\boxed{\sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n} = \quad \text{serie geometrica} \quad r = \frac{1}{4} \Rightarrow \text{converge}$$

$$= \frac{1}{1-r} = \frac{1}{1-\frac{1}{4}} = \frac{1}{\frac{4-1}{4}} = \boxed{\frac{4}{3}}$$

2) $$\sum_{n=1}^{\infty} e^{n/2} = \sum_{n=1}^{\infty} \left(e^{\frac{1}{2}}\right)^n = \sum_{n=1}^{\infty} \left(\sqrt{e}\right)^n \quad - \text{diverge}$$

$$(5^2)^3 = 5^{2 \times 3}$$

serie geom. $r = \sqrt{e} > 1$

diverge

3) $\sum_{n=2}^{\infty} (-1)^{n} 3^{(1-n)} = \sum_{n=2}^{\infty} (-1)^{n} \cdot 3 \cdot 3^{-n} =$

$= 3 \sum_{n=2}^{\infty} (-1)^{n} \cdot 3^{-n} = 3 \sum_{n=2}^{\infty} \frac{(-1)^{n}}{3^{n}} = \boxed{3 \sum_{n=2}^{\infty} \left(-\frac{1}{3}\right)^{n}}$

$\sum_{n=2}^{\infty} \left(-\frac{1}{3}\right)^{n}$ serie geom $r = -\frac{1}{3}$ $-1 < r < 1$ converge

$(*)$ $\left[\begin{array}{l} r_0 = \left(-\frac{1}{3}\right)^{0} = 1 \\ r_1 = \left(-\frac{1}{3}\right)^{1} = -\frac{1}{3} \end{array} \right]$

$= \frac{1}{1-r} - r_0 - r_1 =$

$\sum_{n=0}^{\infty} (\ )^{n} = \frac{1}{1-r}$ $\sum_{n=1}^{\infty} (\ )^{n} = \frac{1}{1-r} - r_0$ $\sum_{n=2}^{\infty} (\ )^{n} = \frac{1}{1-r} - r_0 - r_1$

$$(*) = \cancel{\frac{1}{1+\frac{1}{3}}} - \cancel{1 - \left(-\frac{1}{3}\right)} = \cancel{\frac{19}{12}} \leftarrow 3\left[\frac{1}{1-r} - r_0 - r_1\right]$$

$\frac{1}{4}$ o $\frac{3}{5}$

CRITERI CONVERGENZA $\sum a_n$ $\sum b_n$

A) · Confronto

· $a_n < b_n$ b_n converge $\Rightarrow$ a_n converge

· $a_n > b_n$ b_n diverge $\Rightarrow$ a_n diverge

B) $\lim_{n \to +\infty} \frac{a_n}{b_n} = 0$ b_n converge $\Rightarrow$ a_n converge

a_n diverge $\Rightarrow$ b_n diverge

C) Confronto asintotico: $\lim_{n \to +\infty} \frac{a_n}{b_n} = l$

a_n e b_n hanno lo stesso carattere

D) RAPPORTO

E) RADICE

ES.

$\cdot) \displaystyle\sum_{n=0}^{\infty} \frac{1+e^n}{1+2^n} =$

$$\lim_{n \to +\infty} \frac{1+e^n}{1+2^n} = \lim_{n \to +\infty} \left(\frac{e^n}{2^n}\right) = \lim_{n \to +\infty} \left(\frac{e}{2}\right)^n =$$

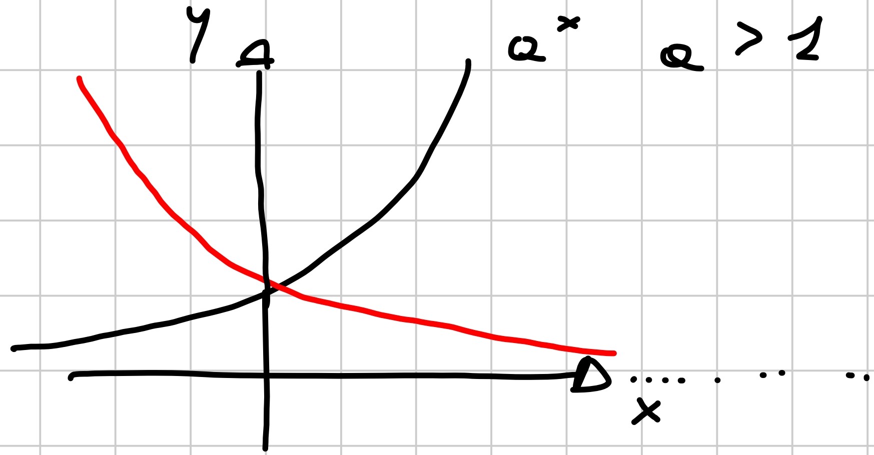
$$= \lim_{n \to +\infty} \left(\frac{e}{2}\right)^n = 0$$

> 1

minore

a^x

$0 < a < 1$



la serie può convergere

- confronto asintotico

$$\frac{1+e^n}{1+2^n} \sim \frac{e^n}{2^n} = \left(\frac{e}{2}\right)^n$$

$n \to +\infty$

converge

$$\sum_{n=0}^{\infty} \left(\frac{e}{2}\right)^n \to \text{converge}$$

serie g. di $r = \frac{e}{2}$

$-1 < r < 1$

$$\lim_{n \to +\infty} \frac{\frac{1+e^n}{1+2^n}}{\frac{e^n}{2^n}} = 1$$

•) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{e^n}$ $\qquad \displaystyle\lim_{n \to +\infty} \frac{n}{e^n} = 0$ può convergere

CONFRONTO

$$\cancel{\begin{array}{l} n > 1 \\ \dfrac{n}{e^n} > \dfrac{1}{e^n} \end{array}}$$

converge

$$\sum \frac{1}{e^n} = \sum \left(\frac{1}{e}\right)^n$$

RAPPORTO

$$\lim_{n\to+\infty} \frac{a_{n+1}}{a_n} = l < 1 \Rightarrow \text{converge}$$

$$\cancel{a_{n+1}} = \quad \sum_{n=0}^{\infty} \frac{n}{e^n}$$

$$a_n = \frac{n}{e^n} \qquad\qquad a_{n+1} = \frac{(n+1)}{e^{n+1}}$$

$$\lim_{n\to+\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to+\infty} \frac{\frac{(n+1)}{e^{n+1}}}{n/e^n} = \lim_{n\to+\infty} \frac{(n+1)}{e^{n+1}} \cdot \frac{e^n}{n}$$

$$= \lim_{n\to+\infty} \frac{(n+1)}{n e^1} = \lim_{n\to+\infty} \frac{\cancel{n}}{\cancel{n} e} = \frac{1}{e} < 1$$

converge

$$\frac{5^2}{5^4} = \frac{1}{5^{4-2}} = \frac{1}{5^2}$$

$$\frac{2^n}{2^{n+1}} = \frac{1}{2^{n+1-1}}$$

$\frac{1}{n^2}$ converge $\qquad$ $\frac{1}{n}$ $\to$ diverge

$\Rightarrow$ $\lim\limits_{n\to+\infty} \dfrac{a_n}{\frac{1}{n^2}} = 0$ $\quad$ a_n converge

$\rightarrow$ $\lim\limits_{n\to+\infty} \dfrac{\frac{1}{n}}{a_n} = 0$ $\quad$ a_n diverge

$$\sum \frac{n}{e^n} \qquad \lim_{n\to+\infty} \frac{\frac{n}{e^n}}{\frac{1}{n^2}} = \lim_{n\to+\infty} \frac{n^3}{e^n} = 0 \quad \text{converge}$$

$\bullet)\ \sum_{n=0}^{\infty} \frac{1}{n+n!}$ $\lim_{n\to+\infty} \left(\frac{1}{n+n!}\right) = 0$ può convergere

$5 > 3$

$\frac{1}{5} < \frac{1}{3}$

$n+n! > n$ NO!

$\frac{1}{n+n!} < \frac{1}{n} \to$ diverge

converge

$n+n! > n!$

$\frac{1}{n+n!} < \frac{1}{n!} \to$ converge

$\sum \frac{1}{n!}$ rapporto $\lim_{n\to+\infty} \frac{a_{n+1}}{a_n} = \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = \frac{n!}{(n+1)!}$

$$= \frac{\cancel{n!}}{(n+1)\cdot \cancel{n!}} = \frac{1}{n+1}$$

$$\boxed{\lim_{n\to+\infty} \frac{1}{n+1} = 0} < 1$$

$$\frac{5!}{6!} = \frac{\cancel{5}\times\cancel{4}\times\cancel{3}\times\cancel{2}\times 1}{6\times\cancel{5}\times\cancel{4}\times\cancel{3}\times\cancel{2}\times 1} = \frac{1}{6}$$

$$= \frac{\cancel{5!}}{6\cdot\cancel{5!}}$$

•) $\displaystyle\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^n$

$\displaystyle\lim_{n\to+\infty} \left(1+\frac{1}{n}\right)^n = e$ la serie diverge

•) $\sum_{n=1}^{\infty} \frac{e^{-n}}{n} = \sum_{n=1}^{\infty} \frac{1}{ne^{n}}$

confronto

converge $\leftarrow$ $\frac{1}{ne^{n}} < \frac{1}{e^{n}}$ $\rightarrow$ converge $\quad n\cdot e^{n} > e^{n}$

$\sum \frac{1}{e^{n}} = \sum \left(\frac{1}{e}\right)^{n}$ serie geometrica converge

confronto $\cancel{\frac{1}{ne^{n}} < \frac{1}{n}}$ $\rightarrow$ diverge $\quad$ NO $\quad n\cdot e^{n} > n$

confronto B) $\lim_{n\to+\infty} \frac{\frac{1}{ne^{n}}}{\frac{1}{n^{2}}} = \lim_{n\to+\infty} \frac{1}{ne^{n}} \cdot n^{2} =$

$$= \lim_{n \to +\infty} \frac{n}{e^n} = 0 \quad \text{OK! converge}$$

Rapporto

$$\lim_{n \uparrow +\infty} \frac{a_{n+1}}{a_n} = \lim_{n \uparrow +\infty} \frac{\frac{1}{(n+1)e^{n+1}}}{\frac{1}{ne^n}} = \lim_{n \uparrow +\infty} \frac{1}{(n+1)e^{n+1}} \cdot ne^n =$$

$$\lim_{n \uparrow +\infty} \frac{ne^n}{(n+1)e^{n+1}} = \lim_{n \uparrow +\infty} \frac{n}{(n+1)e^{1}} = \frac{1}{e} < 1 \Rightarrow \text{converge}$$

$$\cdot) \sum_{n=1}$$

$$= \lim_{n \uparrow +\infty} \frac{n}{ne + \cancel{e}} =$$

$$= \lim_{n \uparrow +\infty} \frac{\cancel{n}}{\cancel{n}e}$$

•) $\sum_{n=2}^{\infty} \frac{1}{\ln n}$

$\ln n < n$

$\frac{1}{\ln n} > \frac{1}{n}$ diverge

diverge

$\lim_{n \to +\infty} \frac{\ln n}{\frac{1}{n^2}} = \lim_{n \to +\infty} \underline{n^2} \cdot \ln n = \infty \Rightarrow$ non converge

$\lim_{n \to +\infty} \frac{\frac{1}{n}}{1/\ln n} = \lim_{n \to +\infty} \frac{\ln n}{n} = 0$ diverge

1) $$\int_0^1 x\sqrt[4]{1+x^2}\,dx = \left[\frac{2}{5}\sqrt[4]{(1+x^2)^5}\right]_0^1 =$$

$$\int x\sqrt[4]{1+x^2}\,dx = \frac{1}{2}\int 2x\,(1+x^2)^{\frac{1}{4}}\,dx = \frac{1}{2}\int f'(x)\,f(x)^{\frac{1}{4}}\,dx$$

$f(x) = 1+x^2$

$f'(x) = 2x$

$$= \frac{1}{2}\,\frac{f(x)^{\frac{1}{4}+1}}{\frac{1}{4}+1} = \frac{1}{2}\,\frac{(1+x^2)^{\frac{5}{4}}}{\frac{5}{4}} = \frac{1}{2}\cdot\frac{4}{5}\sqrt[4]{(1+x^2)^5}$$

$$= \frac{2}{5}\sqrt[4]{(1+x^2)^5}$$

$$= \frac{2}{5}\sqrt[4]{(1+1)^5} - \frac{2}{5}\sqrt[4]{(1+0)^5} = \frac{2}{5}\sqrt[4]{2^5} - \frac{2}{5}$$

$$= \frac{2}{5} \cdot 2\sqrt[4]{2} - \frac{2}{5}$$

$$\cdot) \int_1^{+\infty} \frac{1}{x^3}\,dx = \lim_{a \to +\infty} \int_1^{a} \frac{1}{x^3}\,dx = \lim_{a \to +\infty} \left[-\frac{1}{2x^2} \right]_1^a =$$

$$\int \frac{1}{x^3}\,dx = \int x^{-3}\,dx = \frac{x^{-3+1}}{-3+1} = \frac{x^{-2}}{-2} = -\frac{1}{2x^2}$$

$$= \lim_{a \to +\infty} \left[-\frac{1}{2a^2} - \left(-\frac{1}{2} \right) \right] = \lim_{a \to +\infty} \left[-\frac{1}{2a^2} + \frac{1}{2} \right] = \frac{1}{2}$$

$$0$$